九十五學年第一學期 PHYS2310 電磁學 期中考試題(共兩頁)

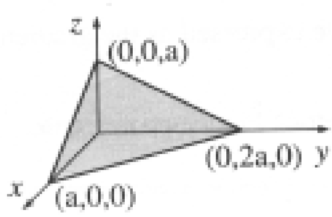
[Griffiths Ch. 1-3] 2006/11/14, 10:10am–12:00am, 教師：張存績

記得寫上學號，班別及姓名等。請依題號順序每頁答一題。

◇ Useful formulas

$$\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\varphi}} \quad \text{and} \quad \nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}v_\phi$$

1. (6%,6%,6%) Show that



(a) $x\frac{d}{dx}(\delta(x)) = -\delta(x)$, [Hint: use integration by parts.]

(b) $\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{\left|\frac{df}{dx}(x_i)\right|}$, where $f(x)$ is assumed to have only simple zero, located at $x = x_i$. [Hint: $\delta(kx) = \frac{1}{|k|}\delta(\mathrm{x})$]

(c) Check Stokes' theorem for the function $\mathbf{v} = y\hat{\mathbf{z}}$, using the triangular shown in the figure.

2. (4%, 4%, 4%, 4%) A metal sphere of radius R, carrying charge q, is surrounded by a thick concentric metal shell (inner radius a, outer radius b). The shell carries no net charge.

(a) Find the surface charge density σ at R, at a, and at b.

(b) Find the electric field at following two regions $R \le r \le a$ and $r \ge b$.

(c) Find the potential at the center ($r = 0$), using the infinity as the reference point.

(d) If the outer shell is grounded, what is the potential of the inner sphere and what is the new surface charge density at b, $\sigma(b)$?

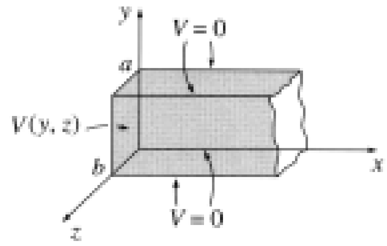
3. (4%, 4%, 8%) The potential of some configuration is given by the expression $V(\mathbf{r}) = Ae^{-\lambda r}/r$, where A and λ are constants.

(a) Find the energy density (energy per unit volume).

(b) Find the charge density $\rho(\mathbf{r})$.

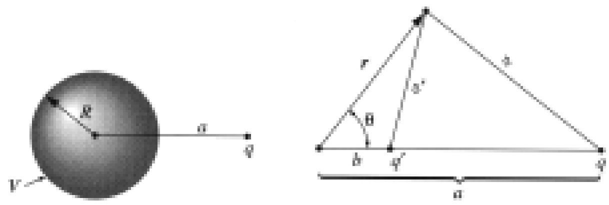
(c) Find the total charge Q (do it two different ways) and verify the divergence theorem.

4. (4%, 6%, 6%) An infinite long rectangular metal pipe (sides *a* and *b*) is grounded, but one end, at *x*=0, is maintained at a specific potential as indicated in the figure, $V(y,z) = V_0 \sin(2\pi y/a)\sin(3\pi z/b)$, where V_0 is a constant.
   (a) Write down the boundary conditions.
   (b) Write down the general solutions.
   (c) Find the potential inside the pipe.



5. (6%, 6%, 6%) Suppose the potential at the surface of a sphere is specified, $V(R_0,\theta) = V_0 \sin^2\theta$, where R_0 is the radius of the sphere and V_0 is a constant. There is no charge inside or outside the sphere.
   (a) Show that the potential outside the sphere.
   (b) Show that the electric field outside the sphere.
   (c) Show that the potential inside the sphere.
   [Hint: use Legendre polynomials, $P_0(x) = 1,\ P_1(x) = x,$ and $P_2(x) = (3x^2 - 1)/2$.]

6. (4%, 6%, 6%) A point charge *q* is situated at distance *a* from the center of a conducting sphere of radius *R*. The sphere is maintained at the constant potential *V*.
   (a) If *V*=0, find the position and value of the image charge.
   (b) Find the electric field on the surface of the metal sphere.
   (c) If $V=V_0$, find the potential outside the sphere.
   [Hint: 1. use the notations shown below. 2. Assume *q* lays on the z-axis]



1.(a)

$$\int_{-\infty}^{\infty} f(x)x\frac{d}{dx}\delta(x)dx = \int_{-\infty}^{\infty} f(x)xd\delta(x)$$

$$= f(x)x\delta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\delta(x)d(f(x)x)$$

$$= -\int_{-\infty}^{\infty}(x\frac{df}{dx}+f)\delta(x)dx$$

$$= -0\frac{df}{dx}\Big|_{x=0} - \int_{-\infty}^{\infty} f(x)\delta(x)dx$$

$$\int_{-\infty}^{\infty} f(x)x\frac{d}{dx}\delta(x)dx = -\int_{-\infty}^{\infty} f(x)\delta(x)dx \quad \Rightarrow \quad x\frac{d}{dx}\delta(x) = -\delta(x)$$

(b)

$$\int_{-\infty}^{\infty} g(x)\delta(f(x))dx = \sum_i \int_{x_i-\varepsilon}^{x_i+\varepsilon} g(x)\delta(\frac{df}{dx}\Big|_{x_i}(x-x_i))dx$$

$$\because \int_{-\infty}^{\infty} g(x)\delta(k(x-x_i))dx = \frac{1}{|k|}g(x_i)$$

$$\Rightarrow \int_{x_i-\varepsilon}^{x_i+\varepsilon} g(x)\delta(\frac{df}{dx}\Big|_{x_i}(x-x_i))dx = \frac{1}{\left|\frac{df}{dx}\Big|_{x_i}\right|}g(x_i)$$

$$\therefore \int_{-\infty}^{\infty} g(x)\delta(f(x))dx = \sum_i \frac{1}{\left|\frac{df}{dx}\Big|_{x_i}\right|}g(x_i) \quad \Rightarrow \quad \delta(f(x)) = \sum_i \frac{1}{\left|\frac{df}{dx}\Big|_{x_i}\right|}\delta(x-x_i)$$

(c)

$$\mathbf{v} = y\hat{\mathbf{z}}, \qquad \nabla\times\mathbf{v} = \hat{\mathbf{x}}, \qquad \int_S (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \int_S \hat{\mathbf{x}}\cdot d\mathbf{a} = \frac{1}{2}a\cdot 2a = a^2$$

(equals the area projected on the $yz-$plane)

$$\oint_P \mathbf{v}\cdot d\mathbf{l} = \int_{\substack{P1 \\ \text{in } zx-\text{plane}}} y\hat{\mathbf{z}}\cdot d\mathbf{l} + \int_{\substack{P2 \\ \text{in } xy-\text{plane}}} y\hat{\mathbf{z}}\cdot d\mathbf{l} + \int_{\substack{P3 \\ \text{in yz}-\text{plane}}} y\hat{\mathbf{z}}\cdot d\mathbf{l}$$

$$= 0 + 0 + \int_{2a}^{0} y(-\tfrac{1}{2}dy) = -\frac{y^2}{4}\Big|_0^{2a} = a^2 \quad \Rightarrow \quad \therefore \int_S (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_P \mathbf{v}\cdot d\mathbf{l}$$

2. Use Gauss's law

(a) $\sigma(R) = \dfrac{q}{4\pi R^2}$, $\sigma(a) = \dfrac{-q}{4\pi a^2}$, and $\sigma(b) = \dfrac{q}{4\pi b^2}$

(b) Use Gauss's law, we obtain $E = \dfrac{q}{4\pi\varepsilon_0 r^2}$ for both two regions, $R \le r \le a$ and $r \ge b$.

(c) $V(r) = -\int_{\infty}^{r}\mathbf{E}\cdot d\mathbf{r}$. $\quad V(0) = V(R) = -\int_{\infty}^{b}\mathbf{E}\cdot d\mathbf{r} - \int_{a}^{R}\mathbf{E}\cdot d\mathbf{r} = \dfrac{q}{4\pi\varepsilon_0}(\dfrac{1}{b}+\dfrac{1}{R}-\dfrac{1}{a})$

(d) $V(r) = -\int_{a}^{r}\mathbf{E}\cdot d\mathbf{r}$, $\quad V(R) = -\int_{a}^{R}\mathbf{E}\cdot d\mathbf{r} = \dfrac{q}{4\pi\varepsilon_0}(\dfrac{1}{R}-\dfrac{1}{a})$ and $\sigma(b) = 0$

3. (a)

$$\mathbf{E} = -\nabla V = -A\frac{\partial}{\partial r}(\frac{e^{-\lambda r}}{r})\hat{\mathbf{r}} = -A\left\{\frac{-\lambda r e^{-\lambda r} - e^{-\lambda r}}{r^2}\right\}\hat{\mathbf{r}} = A\frac{(\lambda r+1)e^{-\lambda r}}{r^2}\hat{\mathbf{r}}$$

$$\text{Energy density} = \frac{\varepsilon_0}{2}E^2 = \frac{\varepsilon_0}{2}A^2\frac{(\lambda r+1)^2 e^{-2\lambda r}}{r^4}$$

(b)

$$\rho = \varepsilon_0\nabla\cdot\mathbf{E} = \varepsilon_0 A(\nabla\cdot\frac{(\lambda r+1)e^{-\lambda r}}{r^2}\hat{\mathbf{r}}) = \varepsilon_0 A(\lambda r+1)e^{-\lambda r}(\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2}) + \varepsilon_0 A\frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla((\lambda r+1)e^{-\lambda r})$$

$$(\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2}) = 4\pi\delta^3(\mathbf{r}) \quad \text{and} \quad (\lambda r+1)e^{-\lambda r}\delta^3(\mathbf{r}) = \delta^3(\mathbf{r})$$

$$\frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla((\lambda r+1)e^{-\lambda r}) = \frac{\hat{\mathbf{r}}}{r^2}\cdot\hat{\mathbf{r}}\frac{\partial}{\partial r}((\lambda r+1)e^{-\lambda r}) = \frac{1}{r^2}\frac{\partial}{\partial r}((\lambda r+1)e^{-\lambda r}) = -\frac{\lambda^2}{r}e^{-\lambda r}$$

$$\rho = \varepsilon_0 A\left[4\pi\delta^3(\mathbf{r}) - \frac{\lambda^2}{r}e^{-\lambda r}\right]$$

(c)

$$Q = \int_v \rho d\tau = \int_v \varepsilon_0 A[4\pi\delta^3(r) - \frac{\lambda^2}{r}e^{-\lambda r}]d\tau = 4\pi\varepsilon_0 A[1 + \int_{r=0}^{\infty}\frac{\lambda^2}{r}e^{-\lambda r}r^2 dr]$$

$$\int_{r=0}^{\infty}\frac{\lambda^2}{r}e^{-\lambda r}r^2 dr = \int_{r=0}^{\infty}e^{-\lambda r}\lambda^2 r dr = -\int_{r=0}^{\infty}\lambda r de^{-\lambda r} = -\int_{x=0}^{\infty}x de^{-x} = -1 \quad \Rightarrow \quad Q = \int_v \rho d\tau = 0$$

Use Gauss's law, the charge enclosed in a sphere of radius R

$$Q_R = \oint_S \varepsilon_0\mathbf{E}\cdot d\mathbf{a} = 4\pi\varepsilon_0 A(\lambda R+1)e^{-\lambda R} \quad \Rightarrow \quad \text{The total charge } Q_{R\to\infty} = 4\pi\varepsilon_0 A(\lambda R+1)e^{-\lambda R}\Big|_{R=\infty} = 0$$

4. (a)

(i) $V = 0$ when $y = 0$,
(ii) $V = 0$ when $y = a$,
(iii) $V = 0$ when $z = 0$,
(iv) $V = 0$ when $z = b$.
(v) $V = V_0\sin(2\pi y/a)\sin(3\pi z/b)$ when $x = 0$,
(vi) $V = 0$ when $x = \infty$

(b)

$$V(x,y,z) = X(x)Y(y)Z(z) \quad \Rightarrow \quad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial y^2} = 0$$

$$\begin{cases} \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2 \quad \Rightarrow \quad Y(y) = A\sin ky + B\cos ky \\ \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -\ell^2 \quad \Rightarrow \quad Z(z) = C\sin\ell z + D\cos\ell z \\ \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2 + \ell^2 \quad \Rightarrow \quad X(x) = Ee^{\sqrt{k^2+\ell^2}x} + Fe^{-\sqrt{k^2+\ell^2}x} \end{cases}$$

$$V(x,y,z) = (A\sin ky + B\cos ky)(C\sin\ell z + D\cos\ell z)(Ee^{\sqrt{k^2+\ell^2}x} + Fe^{-\sqrt{k^2+\ell^2}x})$$

(c)

B.C.(i) $\Rightarrow B = 0$

B.C.(ii) $\Rightarrow \sin ka = 0, \quad k = \dfrac{n\pi}{a} \quad n = 1,2,3,\ldots$

B.C.(iii) $\Rightarrow D = 0$

B.C.(iv) $\Rightarrow \sin \ell b = 0, \quad \ell = \dfrac{m\pi}{b} \quad \mathrm{m} = 1,2,3,\ldots$

B.C.(vi) $\Rightarrow E = 0$

B.C.(v) $\Rightarrow \; V(x,y,z) = \sum_{n,m} ACEe^{-\sqrt{(k^2+\ell^2)}x} \sin(\frac{n\pi}{a}y)\sin(\frac{m\pi}{b}z)$

$$V(0,y,z) = \sum_{n,m} C_{nm} \sin(\frac{n\pi}{a}y)\sin(\frac{m\pi}{b}z) = V_0 \sin(\frac{2\pi}{a}y)\sin(\frac{3\pi}{b}z)$$

$$C_{nm} = \frac{4V_0}{ab}\left[\int_0^a \sin(\frac{2\pi}{a}y)\sin(\frac{n\pi}{a}y)dy\right]\left[\int_0^b \sin(\frac{3\pi}{b}z)\sin(\frac{m\pi}{b}z)dz\right]$$

$C_{23} = V_0, \quad C_{nm} = 0$ when $n \neq 2$ and $m \neq 3$

$$V(x,y) = V_0 e^{-\pi\sqrt{(\frac{2^2}{a}+\frac{3^2}{b})}x} \sin(\frac{2\pi}{a}y)\sin(\frac{3\pi}{b}z)$$

5. (a)

Boundary condition $\begin{cases} \text{(i) } V(R_0,\theta) = V_0 \sin^2\theta \\ \text{(ii) } \lim\limits_{r\to\infty} V(r,\theta) = 0 \end{cases}$

General solution $V(r,\theta) = \sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-(\ell+1)})P_\ell(\cos\theta)$

B.C. (ii) $\rightarrow A_\ell = 0$

B.C. (i) $\rightarrow B_0 R_0^{-1} + B_1 R_0^{-2}\cos\theta + B_2 R_0^{-3}(\frac{3}{2}\cos^2\theta - \frac{1}{2}) = V_0(1-\cos^2\theta)$

$$\begin{cases} B_0 R_0^{-1} - \frac{1}{2}B_2 R_0^{-3} = V_0 \\ \frac{3}{2}B_2 R_0^{-3} = -V_0 \\ B_1 = 0 \end{cases} \Rightarrow \begin{cases} B_0 = \frac{2}{3}R_0 V_0 \\ B_1 = 0 \\ B_2 = -\frac{2}{3}R_0^3 V_0 \end{cases}$$

$$\therefore V(r,\theta) = \frac{2R_0V_0}{3r} - \frac{2R_0^3V_0}{3r^3}P_2(\cos\theta)$$

(b)

$$V(r,\theta) = \frac{2R_0V_0}{3r} - \frac{R_0^3V_0}{3r^3}(3\cos^2\theta - 1)$$

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}} = \left[\frac{2R_0V_0}{3r^2} - \frac{R_0^3V_0}{r^4}(3\cos^2\theta - 1)\right]\hat{\mathbf{r}} - \frac{1}{r}\left[-\frac{R_0^3V_0}{3r^3}(-6\cos\theta\sin\theta)\right]\hat{\boldsymbol{\theta}}$$

$$\mathbf{E} = \left[\frac{2R_0V_0}{3r^2} - \frac{R_0^3V_0}{r^4}(3\cos^2\theta - 1)\right]\hat{\mathbf{r}} - \left[\frac{R_0^3V_0}{r^4}(\sin 2\theta)\right]\hat{\boldsymbol{\theta}}$$

(c)

Boundary condition $\begin{cases} \text{(i) } V(R_0,\theta) = V_0 \sin^2\theta \\ \text{(ii) } \lim_{r\to 0} V(0,\theta) = 0 \end{cases}$

General solution $V(r,\theta) = \sum_{\ell=0}^{\infty} (A_\ell r^\ell + B_\ell r^{-(\ell+1)}) P_\ell(\cos\theta)$

B.C. (ii) $\to B_\ell = 0$

B.C. (i) $\to A_0 + A_1 R_0^1 \cos\theta + A_2 R_0^2 (\frac{3}{2}\cos^2\theta - \frac{1}{2}) = V_0 (1-\cos^2\theta)$

$$\begin{cases} A_0 - \frac{1}{2} A_2 R_0^2 = V_0 \\ \frac{3}{2} A_2 R_0^2 = -V_0 \\ A_1 = 0 \end{cases} \Rightarrow \begin{cases} A_0 = \frac{2}{3} V_0 \\ A_1 = 0 \\ A_2 = -\frac{2}{3} R_0^{-2} V_0 \end{cases}$$

$$\therefore V(r,\theta) = \frac{2V_0}{3} - \frac{2V_0}{3R_0^2} r^2 P_2(\cos\theta)$$

6.(a)

Assume the image charge q' is placed at a distance b from the center of the sphere.
It is equipotential on the surface of a grounded sphere.
Using two boundary conditions at P_1 and P_2

$$\left.\begin{aligned} &\text{At } P_1\text{:} \quad \frac{1}{4\pi\varepsilon_0}\left(\frac{q'}{R-b} + \frac{q}{a-R}\right) = 0 \\ &\text{At } P_2\text{:} \quad \frac{1}{4\pi\varepsilon_0}\left(\frac{q'}{R+b} + \frac{q}{a+R}\right) = 0 \end{aligned}\right\} \text{ two equations and two unknowns } (q' \text{ and } b)$$

$$b = \frac{R^2}{a} \text{(position)}, \quad q' = -\frac{R}{a} q \text{(value of the image charge)}$$

(b)

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q'}{|\mathbf{r}-b\hat{\mathbf{z}}|} + \frac{q}{|\mathbf{r}-a\hat{\mathbf{z}}|}\right\} \text{ where } \begin{cases} |\mathbf{r}-b\hat{\mathbf{z}}| = \sqrt{(r^2\sin^2\theta + (r\cos\theta - b)^2} = \sqrt{(r^2 + 2br\cos\theta - b^2)} \\ |\mathbf{r}-a\hat{\mathbf{z}}| = \sqrt{(r^2\sin^2\theta + (r\cos\theta - a)^2} = \sqrt{(r^2 + 2ar\cos\theta - a^2)} \end{cases},$$

On the surface of the metal sphere, only radial component survives $\mathbf{E} = -\nabla V(\mathbf{r}) = -\frac{\partial V}{\partial r}\hat{\mathbf{r}}$

$$\begin{aligned} \mathbf{E} &= \frac{-1}{4\pi\varepsilon_0}\frac{\partial}{\partial r}\left\{\frac{q'}{\sqrt{(r^2 - 2br\cos\theta + b^2)}} + \frac{q}{\sqrt{(r^2 - 2ar\cos\theta + a^2)}}\right\}\hat{\mathbf{r}} \\ &= \frac{1}{4\pi\varepsilon_0}\left\{\frac{q'(r - b\cos\theta)}{(r^2 - 2br\cos\theta + b^2)^{3/2}} + \frac{q(r - a\cos\theta)}{(r^2 - 2ar\cos\theta + a^2)^{3/2}}\right\}\hat{\mathbf{r}} \end{aligned}$$

(c)

The potential outside the sphere when V=0

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q'}{|\mathbf{r}-b\hat{\mathbf{z}}|} + \frac{q}{|\mathbf{r}-a\hat{\mathbf{z}}|}\right\}, \text{ where } b = \frac{R^2}{a} \text{ and } q' = -\frac{R}{a}q$$

The potential outside the sphere when V=V_0

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{4\pi\varepsilon_0 R V_0}{r} + \frac{q'}{|\mathbf{r}-b\hat{\mathbf{z}}|} + \frac{q}{|\mathbf{r}-a\hat{\mathbf{z}}|}\right\}$$